Lie Groups

A. Kirtland

Outline

Basic Definitions

Starting on Page 150

Definition (Lie group)

A **Lie group** is a smooth manifold w/o boundary that is also a group and the multiplication map $m : G \times G \to G$ and inversion $i : G \to G$ are smooth. Equivalently, the map $(g, h) \mapsto gh^{-1}$ is smooth.

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Remark

A Lie group is also a topological group.

Example: $\mathrm{GL}(n, \mathbb{R})$

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- multiplication smooth b/c entries are polynomials
- inversion by Cramer's rule (gives an algebraic formula for the inverse)

similarly, $\mathrm{GL}(n, \mathbb{C})$ is a dimension $2n^2$ Lie group

Example: An Open Subgroup

- group operations restrict and are smooth

Example: $\mathrm{GL}^+(n, \mathbb{R})$

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- open subset b/c is preimage of $(0, \infty)$ under det, which is continuous
- group operations are restrictions of operations of $\mathrm{GL}(n, \mathbb{R})$ so are smooth

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- $\mathbb{S}^1 \subset \mathbb{C}^*$ is a group under complex multiplication. With appropriate local coordinates, we can write multiplication and inversion as $(\theta_1, \theta_2) \mapsto \theta_1 + \theta_2$ and $\theta \mapsto -\theta$ (Problem 1-8), so this is a Lie group.

Examples: $\mathbb{R}^n$, $\mathbb{C}^n$

(with addition)

$$x - y$$

is linear and so smooth

Example: Direct Product group

$$G_1 \times \cdots \times G_k,\ (g_1, \ldots, g_k)(g_1', \ldots, g_k') = (g_1 g_1', \ldots, g_k g_k')$$

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Example

$\mathbb{T}^n$

Definition (discrete (Lie) group)

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Lie group homomorphisms

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The inclusion $\mathbb{S}^1 \hookrightarrow \mathbb{C}^*$

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- exp : $(\mathbb{C}, +) \to (\mathbb{C}^*, *)$ is surjective but not injective.

Example: ε

$$\varepsilon : \mathbb{R} \to \mathbb{S}^1, t \mapsto e^{2\pi it}$$

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$$\varepsilon^n : \mathbb{R}^n \to \mathbb{T}^n$$

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Example: det

$$\mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^*$$

$$\mathrm{GL}(n, \mathbb{C}) \to \mathbb{C}^*$$

are smooth b/c it is a polynomial of matrix entries of input, and is a homomorphism b/c $\det AB = \det A \det B$.

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conjugation $C_g : G \to G, h \mapsto ghg^{-1}$ is smooth b/c multiplcation and inversion are smooth. This gives a Lie isomorphism as it has inverse $C_{g^{-1}}$.

Definition: Translation Maps

The translation maps $L_g, R_g : G \to G$ are diffeomorphisms of G with inverses L_{g^1} and $R_{g^{-1}}$.

$$L_g(h) = gh$$

$$R_g(h) = hg$$

are smooth because they are the composition of smooth maps $G \xrightarrow{\iota_g} G \times G \xrightarrow{m} G$ where $\iota_g(h) = (g, h)$ and m is multiplication.

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are smooth because they are the composition of smooth maps $G \xrightarrow{\iota_g} G \times G \xrightarrow{m} G$ where $\iota_g(h) = (g, h)$ and m is multiplication. The translation maps are smooth, transitive group actions of G on itself.

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- L_k for any k is a diffeomorphism so the $d(L_k)$ maps above are linear isomorphisms.
- Composing a linear map with a linear isomorphism does not change the rank, so $\operatorname{rank} dF_{g_0} = \operatorname{rank} dF_e$

Theorem: A Lie group homomorphism is a Lie group isomorphism iff it is bijective.

The global rank theorem says that b/c F is bijective and has constant rank, it is a diffeomorphism, and this inverse is a Lie homomorphism by definition.

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Lifted multiplication and inversion are smooth. (2)

- Exercise: Let M, N, P be smooth manifolds, $F : M \to N$ be a local diffeomorphism. If $G : P \to M$ is continuous, then G is smooth iff $F \circ G$ is smooth.

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$$\begin{array}{ccc} \tilde{G} & \xrightarrow{\tilde{i}} & \tilde{G} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G & \xrightarrow[i]{} & G \end{array}$$

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- similar proof shows $x\tilde{e} = x$

(3) Multiplication in $\tilde{G}$ is associative.

- Let $\alpha_L, \alpha_R : \tilde{G} \times \tilde{G} \times \tilde{G} \to \tilde{G}$

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- Both α_L and α_R are lifts of $\alpha(x, y, z) = \pi(x)\pi(y)\pi(z)$ because

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- So $\tilde{G}$ is a group!
- Problem 7-5: Show that the universal covering group is unique up to Lie group isomorphism.

Lie subgroups

Definition: Lie subgroup

A subgroup of G with a topology and smooth structure that makes it an immersed submanifold of G.

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- Want to show multiplication and inversion are smooth.
- Recall 5.30, a smooth map to an manifold whose image is contained in an embedded submanifold is smooth as a map to the submaifold.
- Multiplication is smooth as a map $H \times H \to G$ and has image contained in H, so it is also smooth as a map into H

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- So $G - H$ is open, H is closed, and because H is open and closed it must be a union of components

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Exercise

Show this subgroup is the set of all elements that can be written as finite products of elements of S and their inverses.

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- W_k is open because for all g, L_g is a diffeomorphism
- By the exercise mentioned before, $H = \cup_{k=1}^{\infty} W_k$, so is open.

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H is also closed as shown before, so if G is connected, then $H = G$.

The identity component

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- Problem 7-7: G_0 is a normal subgroup of G and is the only connected open subgroup. Every connected component of G is diffeomorphic to G_0.

Theorem: The kernel of a Lie group homomorphism is a properly embedded Lie subgroup.

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- Applying this to $\ker F = F^{-1}(\tilde{e})$ and using that fact that $\ker F$ is a subgroup gives us that it is a Lie subgroup

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Examples: Embedded Lie subgroups of $GL(n, \mathbb{R})$

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Example

$\beta : \mathrm{GL}(n,\mathbb{C}) \to \mathrm{GL}(2n,\mathbb{R})$ defined by replacing each entry $a + ib$ with the block $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ We can check that β is an injective Lie group homomorphism whose image is a properly embedded subgroup of $\mathrm{GL}(2n,\mathbb{R})$.

Examples with $\mathbb{C}$

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$\mathbb{S}^1 \subset \mathbb{C}^*$ as it is a subgroup and an embedded submanifold

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Example

$\mathrm{SL}(n, \mathbb{C}) \subset \mathrm{GL}(n, \mathbb{C})$ just as with the real case, but with dimension $2n^2 - 2$.

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- For V a real or complex vector space, GL(V) is a group under composition.
- Any finite basis for V determines an isomorphism with GL($n, \mathbb{R}$) or GL($n, \mathbb{C}$), so it's a Lie group
- The transition map between any two such isomorphisms is of the form $A \mapsto ABA^{-1}$, which is smooth, so there is a diffeomorphism between them and the manifold structure on GL(V) is coordinate-independent

Example: non-embedded Lie subgroup

The dense curve in the Torus $\gamma : \mathbb{R} \to \mathbb{T}^2$ is an immersed Lie subgroup b/c γ is an injective Lie homomorphism.

The Lie Algebra of a Lie Group

Lie Bracket

The Lie bracket of two vector fields X, Y is

$$[X, Y]\, f = XYf - YXf$$

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- For F a diffeomorphism, $F_* [X, Y] = [F_* X, F_* Y]$

Definition

A vector field X on G is left-invariant if it is invariant under all left translations, or it is L_g-related to itself for all g.

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Because L_g is a diffeomorphism, $(L_g)_*X$ is defined as $((L_g)_*X)_q = d(L_g)_{(L_g)^{-1}(q)}(X_{(L_g)^{-1}(q)})$, which gives us the equivalent condition

$$(L_g)_*X = X$$

the set of left-invariant vector fields is closed under the Lie bracket

$$(L_g)_*[X, Y] = [(L_g)_* X, (L_g)_* Y] = [X, Y]$$

Properties of the Lie Bracket

- Bilinear: for $a, b \in \mathbb{R}$, $[aX + bY, Z] = a[X, Z] + b[Y, Z]$

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- Bilinear: for $a, b \in \mathbb{R}$, $[aX + bY, Z] = a[X, Z] + b[Y, Z]$
- Antisymmetric: $[X, Y] = -[Y, X]$
- Jacobi Identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

Definition: Lie algebra

A Lie algebra over $\mathbb{R}$ is a real vector space $\mathfrak{g}$ with a bracket from $\mathfrak{g} \times \mathfrak{g}$ to $\mathfrak{g}$ denoted $(X, Y) \mapsto [X, Y]$ that is bilinear, antisymmetric, and satisfies the Jacobi identity.

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Remark

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A Lie subalgebra is a linear subspace that is closed under the bracket.

Example

For a Lie group, $\mathrm{Lie}(G)$, the set of all left-invariant smooth vector fields, is a Lie subalgebra of $\mathfrak{X}(G)$ (which is a Lie algebra no matter what).

Examples: General vector space

- $\mathfrak{gl}(V)$, the set of (not necessarily invertible) linear maps from V to itself is a Lie algebra under the commutator bracket $[A, B] = A \circ B - B \circ A$.

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- $\mathfrak{gl}(V)$, the set of (not necessarily invertible) linear maps from V to itself is a Lie algebra under the commutator bracket $[A, B] = A \circ B - B \circ A$.
- any vector space can be made into an abelian Lie algebra with the zero bracket